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Mathematical Construction of Chiral Anomaly

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Chiral anomaly is constructed with mathematical rigor by means of the lattice regularization. To this end, the quantized Dirac fermion on a torus coupled with an external gauge field with an arbitrary topology is analyzed. It is claimed that in order to complete the rigorous lattice approach to the chiral anomaly, the heat kernel regularization for the integrand should be introduced independently of the lattice regularization for the path measure.

1. INTRODUCTION

Chiral anomaly is the breakdown of the classical conservation law for the chiral current of a fermion due to the quantum effect. This phenomenon has been subjected to extensive investigations from several viewpoints: perturbative field theoretical [1–3], functional integral [4, 5], and differential geometric [6, 7], etc. Through such works, it has been recognized that the chiral anomaly is deeply connected with the Atiyah–Singer index theorem [8–10].

Furthermore, in order to clarify mathematical obscurity associated with the quantization procedure, the constructive approach based on the lattice regularization has been pursued by several workers [11–18]. They however used some expansion techniques which are not mathematically rigorous. Although they seem to have regarded the remaining problem as a "mathematical detail," no one has rigorously completed their approach. We have to suspect that something is yet lacking.

Needless to say, the lattice regularization is designed to define the *path measure*. But the lattice cutoff automatically regularizes the *integrand* as well. This might seem to be a convenience, but we must be more careful of the integrand, because there is no a priori reason for the optimism that the prescription defining the path measure will necessarily be suitable for the description of the delicate cancellation of the integrand as a result of which the chiral anomaly arises.

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On the other hand, it was proposed that the heat kernel regularization be introduced into the constructive field theory with a flavor of the index theorem [19, 20]. Such works suggest that the heat kernel effectively works as a regulator for the integrand for which the infinite-dimensional integration does not absolutely converge.

In this paper, besides the lattice regularization of the path measure, we introduce the heat kernel regularization for the integrand (see Subsection 2.3) and complete the rigorous lattice approach to the chiral anomaly. This procedure considerably simplifies the argument on the continuum limit by separating the independent problems: the definition of path measure and the integration of the functional that is not absolutely convergent. At the same time, it yields a mathematical foundation for the intuitive picture of the chiral anomaly.

Let us explain the picture. As is well known, in order to suppress the species doubling and to obtain the correct anomaly in the continuum limit, Wilson [21] proposed adding some chiral symmetry breaking terms to the naive lattice action of the fermion (see also [13, 16, 22, 23]). (The doubled species are eigenfunctions of the lattice (free) Dirac operator with very large momenta (approximately equal to the inverse lattice spacing) but small energies.) Note that, in the continuum limit, the doubled species converge weakly to zero. They seem to go out of the L^2 space, bearing the chiral current. This is the origin of the chiral anomaly and is the reason why we must suppress the species doubling in the lattice theory by means of the chiral symmetry breaking terms.

To summarize, we define the notion of doubled species in *a nonperturbative way* and prove that, in the continuum limit,

(1) the doubled species go out of the L^2 space;

(2) the L^2 space is spanned by the remaining eigenfunctions;

and

(3) the Wilson term [16] works as "the projection" onto the L^2 space.

By realizing the above picture, we take the continuum limit. As a result, we arrive at "the left-hand side of the (local) index theorem" [24, 25]. Our end is just the right-hand side of it. Furthermore we obtain a lattice theoretical picture of the index of Dirac operator, in which an interesting relation is found between the "metaphysical" concept of index and the "physical" procedure of discretization.

In Section 2, we formulate the problem and the result. In Section 3, we construct the projection eliminating the doubled species. Section 4 is devoted to the analysis on the spectrum of the lattice Dirac operator. The continuum limit is taken in Section 5.

2. Formulation

Based on the lattice regularization, we analyze the fermion on the d (even)-dimensional torus interacting with an external (classical) gauge field with an arbitrary topology in the Euclidean framework.

2.1. *Dirac Operator*

Let E_0 be a complex vector bundle with a gauge group G over the even-dimensional torus $\Lambda_0 = (\mathbb{R}/L\mathbb{Z})^d$, $L \in \mathbb{Z}_+$, and fix a fiber metric and a metric connection ∇_0 on it. We introduce the trivial spin structure by making the tensor product $E = \Delta \otimes E_0$ with the trivial spinor bundle $\Delta = \Lambda_0 \times \mathbb{C}^{2^{d/2}}$ over Λ_0. Let ∇ be the resulting connection of E and let $\bar{\not{D}}$ denote the Dirac operator made from ∇. It is locally written as

$$\bar{\not{D}} = \sum_{\mu=1}^{d} \gamma^\mu \otimes \left(I \frac{\partial}{\partial x^\mu} + iA_\mu(x) \right),$$

or, for simplicity,

$$\bar{\not{D}} = \sum_{\mu=1}^{d} \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + iA_\mu(x) \right),$$

where γ^μ, $\mu = 1, 2, ..., d$, are the antihermitian Dirac matrices in the Euclidean framework and $A_\mu(x)$, $\mu = 1, 2, ..., d$, turn out to be hermitian.

Let $L^2(\Lambda_0)$ be the Hilbert space of all L^2 sections of E with the inner product

$$(u, v) = \int_{\Lambda_0} (u(x), v(x))_x \, dx,$$

where $(\ ,\ )_x$ is the fiber metric of E at $x \in \Lambda_0$. The Dirac operator $\bar{\not{D}}$ is self-adjoint on $L^2(\Lambda_0)$ and has a pure point spectrum accumulating only at ∞, and its eigenspaces are finite dimensional. Furthermore, the eigenfunctions are smooth [26].

2.2. *Lattice Fermion*

We now proceed to define the lattice Dirac operator on the discrete torus $\Lambda_a = (a\mathbb{Z}/L\mathbb{Z})^d$, where $a = 2^{-l}$, $l \in \mathbb{Z}_+$. We regard the space of all sections of $E|_{\Lambda_a}$ as a (finite-dimensional) Hilbert space with the inner product

$$(u, v) = a^d \sum_{x \in \Lambda_a} (u(x), v(x))_x$$

and denote this Hilbert space by $L^2(\Lambda_a)$.

Let T_μ, $\mu = 1, 2, ..., d$, be the covariant translation by the lattice spacing a along the μth axis with respect to the connection ∇_0 and put

$$\not{D} = (2a)^{-1} \sum_{\mu=1}^{d} \gamma^\mu (T_\mu - T_\mu^{-1}). \tag{2.1}$$

Note that T_μ is unitary and $\not{D}$ is hermitian on $L^2(\Lambda_a)$.

The lattice action of the fermion [16] is given by

$$S_a(\Psi', \Psi) = \left(\Psi', \left(i\not{D} + MI - \frac{r}{2a} e^{i\theta\gamma^5} W \right) \Psi \right),$$

where r, $M > 0$, $|\theta| < \pi/2$, $\gamma^5 = -i^{d/2}\gamma^1\gamma^2 \cdots \gamma^d$, and

$$W = \sum_{\mu=1}^{d} (T_\mu + T_\mu^{-1} - 2I).$$

Then, the vacuum expectation is defined by the Berezin integral [9, 27]

$$\langle \cdot \rangle_a = Z_a^{-1} \int d\Psi' \, d\Psi \cdot \exp(-S_a(\Psi', \Psi)),$$

where

$$Z_a = \int d\Psi' \, d\Psi \exp(-S_a(\Psi', \Psi)).$$

Let us consider the chiral current [16]

$$J^{5\mu}(x) = \tfrac{1}{2}(\Psi'_x, \gamma^5\gamma^\mu (T_\mu \Psi)_x)_x + \tfrac{1}{2}((T_\mu \Psi')_x, \gamma^5\gamma^\mu \Psi_x)_x, \quad \mu = 1, 2, ..., d,$$

and put

$$Y(x) = \frac{1}{a} \sum_{\mu=1}^{d} (J^{5\mu}(x) - J^{5\mu}(x - a\hat{\mu})) - 2Mi(\Psi'_x, \gamma^5 \Psi_x)_x$$

for $x \in \Lambda_a$. As is easily seen, it holds that

$$\begin{aligned} Y(x) = {} & (\Psi'_x, \gamma^5 (\not{D}\Psi)_x)_x + ((\not{D}\Psi')_x, \gamma^5 \Psi_x)_x \\ & - 2Mi(\Psi'_x, \gamma^5 \Psi_x)_x, \qquad x \in \Lambda_a. \end{aligned} \tag{2.2}$$

2.3. *Heat Kernel Regularization*

Since the expectation $\langle Y(x) \rangle_a$ is well defined on the lattice, one would think that the problem is to compute $\lim_{a \to 0} \langle Y(x) \rangle_a$. As a matter of fact, we furthermore must regularize the integrand $Y(x)$ by means of the heat kernel, since γ^5 does not have a well-defined trace as an operator on $L^2(\Lambda_0)$. Namely, we regularize γ^5 as $\gamma^5 e^{-t\not{D}^2}$, $t>0$, and put

$$Y_t(x) = (\Psi'_x, \gamma^5(e^{-t\not{D}^2}\not{D}\Psi)_x)_x + ((\not{D}\Psi')_x, \gamma^5(e^{-t\not{D}^2}\Psi)_x)_x \\ - 2Mi(Y'_x, \gamma^5(e^{-t\not{D}^2}\Psi)_x)_x, \qquad x \in \Lambda_a,$$

Our problem is to calculate $\lim_{t\to 0} \lim_{a\to 0} \langle Y_t(x) \rangle_a$.

Remarks. 1. The introduction of the heat kernel regularization may seem to be redundant, since $\langle Y(x)\rangle_a$ is well defined. However, when we define a singular integral $\int_{-1}^{1} f(x)\,dx$ for a function $f(x)$ with a singularity at $x=0$, we shall use in addition some prescription like Cauchy's principal value, even if we could make shift with the Riemann sum. Namely, we prefer to gain simplicity by separating independent problems rather than by saving prescriptions. Introducing the heat kernel regularization, we can separate the problem of integrating the functional that is not absolutely integrable from the problem of defining the path measure by using the lattice regularization. Moreover, the heat kernel regularization for the integrand can be removed by means of the index theorem (see Section 5). This is the advantage of our procedure.

2. We do not know whether $\lim_{a\to 0} \langle Y(x)\rangle_a$ exists or not, nor do we have numerical evidence. In this situation, we are in favor of the conjecture that there exists a gauge field configuration which does not admit $\lim_{a\to 0} \langle Y(x)\rangle_a$.

3. A reader who respects the symmetry principles embodied in the lattice Ward identity [16] should note that the heat-regularized chiral current is derived from the heat-regularized chiral transformation

$$\Psi \mapsto \exp\left(i\frac{\alpha}{2}\gamma^5 e^{-t\not{D}^2}\right)\Psi, \qquad \Psi' \mapsto \exp\left(-i\frac{\alpha}{2}\gamma^5 e^{-t\not{D}^2}\right)\Psi',$$

though the resulting Ward identity will be of no use to us.

2.4 *Result.*

In the subsequent sections, we show the equality

$$\lim_{t\to 0}\lim_{a\to 0} \langle Y_t(x)\rangle_a = 2\,\mathrm{ch}_{d/2}(F),$$

where F stands for the curvature of the gauge field and ch denotes the Chern class. Precisely speaking, we prove the following:

THEOREM. *For any smooth function $\xi(x)$ on the continuum torus Λ_0, it holds that*

$$\lim_{t\to 0}\lim_{a\to 0} a^d \sum_{x\in\Lambda_a} \xi(x)\langle Y_t(x)\rangle_a$$
$$= -2i\int_{\Lambda_0} dx\, \xi(x)\left(\frac{1}{4\pi}\right)^{d/2} \frac{1}{(d/2)!} \sum_{\alpha_1,\dots,\alpha_d=1} \varepsilon^{\alpha_1\alpha_2\cdots\alpha_d}$$
$$\times \operatorname{tr}(F_{\alpha_1\alpha_2}(x)\, F_{\alpha_3\alpha_4}(x)\cdots F_{\alpha_{d-1}\alpha_d}(x)),$$

where $\varepsilon^{\alpha_1\alpha_2\cdots\alpha_d}$ is the completely antisymmetric tensor.

Remark. Note that Ind $\not{D}$ always vanishes whether Ind $\bar{\not{D}}=0$ or not. Our solution to this paradox is the equality

$$\operatorname{Ind}\bar{\not{D}} = \operatorname{Ind}\hat{\not{D}}(\lambda_*),$$

which is shown in Section 4 for any $\lambda_*>0$ and for a sufficiently small $a>0$. Here, $\hat{\not{D}}(\lambda_*)$ is defined by (3.19) as the "restriction" of the lattice Dirac operator $\not{D}$ to the low energy ($|\lambda|\leqslant\lambda_*$) eigenspace *without doubling modes.* Namely, the dimensions of Ker $\bar{\not{D}}$ are partially lost as the doubling modes and this fact causes the upset of chiral balance in the kernel when the topology of the gauge field is nontrivial.

3. SPECIES DOUBLING

The operator T_μ^2 acts on low energy spinor fields as an approximate identity (see Lemma 3.1(iii)), that is, $T_\mu \approx \pm 1$ holds in the low energy region. This allows us to identify doubled species with the low energy eigenfunctions for which $T_\mu \approx -1$ for some $\mu = 1, 2, \dots, d$. In this section, we construct the projection eliminating the doubled species in a nonperturbative way (see (3.18)).

Let

$$\not{D} = \sum_n \lambda_n P_n$$

be the spectral decomposition, where the λ_n's are eigenvalues of $\not{D}$. We arbitrarily choose a positive number λ_* that does not belong to the spectrum of $\not{D}$ and put

$$P(\lambda_*) = \sum_{n:|\lambda_n|\leqslant\lambda_*} P_n \tag{3.1}$$

$$\Delta = \min_n |\lambda_n - \lambda_*| \tag{3.2}$$

$$N = \dim \operatorname{Im} P(\lambda_*). \tag{3.3}$$

(When we take the continuum limit $a \to 0$ in the subsequent sections, we shall vary λ_* depending on a, while in this section we arbitrarily fix a and λ_* so that $\Delta \neq 0$.) Note that $P(\lambda_*)$ commutes with γ^5 since the operator $\not{D}$ anti-commutes with γ^5.

In what follows, the constants C_j can be chosen independently of a, t, and λ_*.

LEMMA 3.1. (i) *For μ, $\nu = 1, 2, ..., d$, the following estimates hold*:

$$\|[T_\mu, T_\nu]\| < C_1 a^2,$$

$$\|[T_\mu, \not{D}]\| < C_2 a.$$

(ii) *Put*

$$J = \frac{1}{8a^2} \sum_{\mu \neq \nu} \gamma^\mu \gamma^\nu [T_\mu - T_\mu^{-1}, T_\nu - T_\nu^{-1}].$$

Then, we have

$$-\sum_{\mu=1}^{d} (T_\mu - T_\mu^{-1})^2 = 4a^2(\not{D}^2 - J),$$

$$\|J\| < C_3.$$

(iii) *For $\mu = 1, 2, ..., d$, and for $u \in L^2(\Lambda_a)$, it holds that*

$$\|(T_\mu - T_\mu^{-1})u\| < C_4 a(\|u\| + \|\not{D}u\|). \tag{3.4}$$

Proof. Part (i) is trivial. Parts (ii) and (iii) follow from (i) and (ii), resp. ∎

The inequality (3.4) shows that T_μ^{-1} is close to T_μ in the low energy region. Considering the combination $\frac{1}{2}(T_\mu + T_\mu^{-1})$ (instead of T_μ itself), we can improve the above estimates.

LEMMA 3.2. (i) *For μ, $\nu = 1, 2, ..., d$, and for $u \in L^2(\Lambda_a)$, the following estimates hold*:

$$\|[T_\mu + T_\mu^{-1}, T_\nu]u\| < C_5 a^3(\|u\| + \|\not{D}u\|), \tag{3.5}$$

$$\|[T_\mu + T_\mu^{-1}, \not{D}]u\| < C_6 a^2(\|u\| + \|\not{D}u\|), \tag{3.6}$$

$$\|[W, \not{D}]u\| < C_7 a^2(\|u\| + \|\not{D}u\|). \tag{3.7}$$

(ii) *For $\mu = 1, 2, ..., d$, it holds that*

$$\|[T_\mu + T_\mu^{-1}, P(\lambda_*)]\| < C_8 a^2 N\Delta^{-1}(1+\lambda_*), \tag{3.8}$$

$$\|[W, P(\lambda_*)]\| < C_9 a^2 N\Delta^{-1}(1+\lambda_*), \tag{3.9}$$

$$\begin{aligned}\|(I-P(\lambda_*))\, WP(\lambda_*)\| &= \|P(\lambda_*)\, W(I-P(\lambda_*))\| \\ &< C_{10} a^2 N\Delta^{-1}(1+\lambda_*).\end{aligned} \tag{3.10}$$

Proof. (i) The $O(a^2)$ term of $[T_\mu, T_\nu]$ cancels with that of $[T_\mu^{-1}, T_\nu]$. Then, we obtain (3.5) by the help of (3.4). The estimates (3.6) and (3.7) follow from (3.5).

(ii) Putting $P = P(\lambda_*)$ and $Q = I - P$, we have, for any matrix B,

$$QBP = \sum_{n:|\lambda_n| \leqslant \lambda_*} (\not{D} - \lambda_n I)^{-1} Q[\not{D}, B]\, P_n,$$

and hence

$$\|QBP\| \leqslant \Delta^{-1} \sum_{n:|\lambda_n| \leqslant \lambda_*} \|[\not{D}, B]\, P_n\|.$$

Furthermore, if $B^* = B$, we have

$$\|PBQ\| = \|QBP\|$$

$$\|[B, P]\| \leqslant 2\, \|QBP\|.$$

Then, (3.9) and (3.10) follow from (3.7), while (3.8) follows from (3.6). ∎

Put

$$S_\mu = \tfrac{1}{2} P(\lambda_*)(T_\mu + T_\mu^{-1})\, P(\lambda_*), \qquad \mu = 1, 2, ..., d. \tag{3.11}$$

LEMMA 3.3. *For $\mu, \nu = 1, 2, ..., d$, we have the estimates*

$$\|S_\mu^2 - P(\lambda_*)\| < C_{11} a^2 N(1+\Delta^{-1})(1+\lambda_*)^2, \tag{3.12}$$

$$\|[S_\mu, S_\nu]\| < C_{12} a^2 N(1+\Delta^{-1})(1+\lambda_*), \tag{3.13}$$

$$\|[S_\mu, \not{D}]\| < C_{13} a^2 (1+\lambda_*). \tag{3.14}$$

Proof. Since

$$\begin{aligned} S_\mu^2 - P(\lambda_*) &= \tfrac{1}{4} P(\lambda_*)(T_\mu - T_\mu^{-1})^2 P(\lambda_*) \\ &\quad + \tfrac{1}{4} P(\lambda_*)[T_\mu + T_\mu^{-1}, P(\lambda_*)](T_\mu + T_\mu^{-1})\, P(\lambda_*), \end{aligned}$$

(3.12) follows from (3.4) and (3.8), and (3.13) and (3.14) follow from (3.5), (3.6), and (3.8). ∎

Furthermore, we modify the S_μ's so that they may commute with each other.

LEMMA 3.4. *If* $a < C_{14} N^{-1/2}(1+\Delta^{-1})^{-1/2}(1+\lambda_*)^{-1}$, *there exist projections* P_μ^+, $\mu = 1, 2, ..., d$, *such that, for* μ, $\nu = 1, 2, ..., d$,

$$[P_\mu^\pm, P_\nu^\pm] = [P_\mu^+, P_\nu^-] = 0, \tag{3.15}$$

$$\|P_\mu^+ - P_\mu^- - S_\mu\| < C_{15} a^2 N^d (1+\Delta^{-1})(1+\lambda_*)^2, \tag{3.16}$$

$$\|[P_\mu^\pm, \not{D}]\| < C_{16} a^2 N^{d+1} (1+\Delta^{-1})(1+\lambda_*)^3, \tag{3.17}$$

where P_μ^- *are defined by*

$$P(\lambda_*) = P_\mu^+ + P_\mu^-.$$

Proof. When a is small, the bound (3.12) implies that the eigenvalues of S_1 are classified into three groups: near ± 1 and 0. We denote by $P_1^\pm$ the projection onto the sum of eigenspaces with the eigenvalues near ± 1, respectively. Then it holds that

$$P(\lambda_*) = P_1^+ + P_1^-,$$

$$\|P_1^+ - P_1^- - S_1\| < C_{17} a^2 N (1+\Delta^{-1})(1+\lambda_*)^2,$$

$$\|[P_1^\pm, S_\mu]\| < C_{18} a^2 N^2 (1+\Delta^{-1})(1+\lambda_*)^3, \qquad \mu = 1, 2, ..., d.$$

The last inequality follows from (3.13) by the same argument as that in Lemma 3.2(ii). Put

$$\tilde{S}_2 = \sum_{\sigma = \pm} P_1^\sigma S_2 P_1^\sigma.$$

Since

$$[P_1^\pm, \tilde{S}_2] = 0,$$

$$\|\tilde{S}_2^2 - P(\lambda_*)\| < C_{19} a^2 N^2 (1+\Delta^{-1})(1+\lambda_*)^2$$

there exist projections $P_2^\pm$ such that

$$[P_\mu^\sigma, P_\nu^\tau] = 0, \qquad \sigma, \tau = \pm, \quad \mu, \nu = 1, 2,$$

$$P(\lambda_*) = P_2^+ + P_2^-,$$

$$\|P_2^+ - P_2^- - \tilde{S}_2\| < C_{20} a^2 N^2 (1+\Delta^{-1})(1+\lambda_*)^2.$$

Put

$$\tilde{S}_3 = \sum_{\sigma, \tau = \pm} P_1^\sigma P_2^\tau S_3 P_1^\sigma P_2^\tau.$$

Then, we can define $P_3^\pm$ just as above. In the same way, the projections $P_\mu^\pm$, $\mu = 3, 4, ..., d$, are successively defined. Equation (3.16) follows directly.

Furthermore, since

$$[P_\mu^+ - P_\mu^-, \not{D}] = [S_\mu, \not{D}] + [P_\mu^+ - P_\mu^- - S_\mu, P(\lambda_*)\not{D}],$$

(3.14) and (3.16) imply

$$\|[P_\mu^+ - P_\mu^-, \not{D}]\| < C_{21} a^2 N^{d+1}(1+\Delta^{-1})(1+\lambda_*)^3.$$

Then we obtain (3.17) from (3.14) and (3.16) by the same argument as the proof of Lemma 3.2(ii). ∎

Using the commutative projections P_μ^+, $\mu = 1, 2, ..., d$, in Lemma 3.4, we put

$$P^+(\lambda_*) = \prod_{\mu=1}^{d} P_\mu^+, \tag{3.18}$$

$$\hat{\not{D}}(\lambda_*) = P^+(\lambda_*)\, \not{D}|_{\mathrm{Im}\, P^+(\lambda_*)}. \tag{3.19}$$

We can regard $P^+(\lambda_*)$ as the projection that eliminates doubled species.

4. Spectra

In this section, we are concerned with the spectra of the operators $\not{D}$ and $\hat{\not{D}}(\lambda_*)$ (see (3.19)) and show Propositions 4.1 and 4.2 below.

Proposition 4.1. *For each $t > 0$, we have*

$$\mathrm{Tr}_a e^{-t\not{D}^2} < \chi(t), \qquad 0 < a < 1,$$

where $\chi(t)$ is independent of a.

Let $\square_x$ be the box $\{y \in \Lambda_0 \mid x^\mu \leqslant y^\mu \leqslant x^\mu + a\}$ and fix, for each $x \in \Lambda_a$, a local trivialization of $E|_{\square_x}$. Since $u \in L^2(\Lambda_a)$ is extended to $\bar{u} \in L^2(\Lambda_0)$ as a constant in each box through the local trivialization, we can regard $L^2(\Lambda_a)$ as a subspace of $L^2(\Lambda_0)$. Thus, we denote the norms in $L^2(\Lambda_a)$ and in $L^2(\Lambda_0)$ by the same notation $\|\ \|$.

The following proposition shows that there exists a one-to-one correspondence between eigenfunctions of $\hat{\not{D}}(\lambda_*)$ and low energy eigenfunction of $\bar{\not{D}}$.

Proposition 4.2. *Let λ_{**} be an arbitrary positive number and let $a > 0$ be sufficiently small for the fixed λ_{**}. Then we can choose $\lambda_*(a) \in [\lambda_{**}, 2\lambda_{**}]$ so that the following statements turn out to be valid:*

(i) *The numbers* $\Delta = \Delta(a)$ *and* $N = N(a)$ *defined by* (3.2) *and* (3.3) (*with* λ_* *replaced by* $\lambda_*(a)$), *respectively, satisfy*

$$\Delta > C_{22}(\lambda_{**}), \tag{4.1}$$

$$N < C_{23}(\lambda_{**}). \tag{4.2}$$

(ii) *Each eigenvalue of* $\hat{\not{D}}(\lambda_*(a))$ *belongs to one of the disjoint intervals* $I_a(\bar{\lambda}) = [\bar{\lambda} - C_{24}a, \bar{\lambda} + C_{24}a]$, *where* $\bar{\lambda}$ *is an eigenvalue of* $\bar{\not{D}}$ *such that* $|\bar{\lambda}| \leqq \lambda_{**}$. *Furthermore, for each eigenvalue* $\bar{\lambda}$ *of* $\bar{\not{D}}$ *such that* $|\bar{\lambda}| \leqq \lambda_{**}$, *the number of eigenvalues of* $\hat{\not{D}}(\lambda_*(a))$ *contained in* $I_a(\bar{\lambda})$ *is equal to the multiplicity of* $\bar{\lambda}$.

(iii) *Let* $\bar{\lambda} \in [-\lambda_{**}, \lambda_{**}]$ *be an eigenvalue of* $\bar{\not{D}}$ *with multiplicity* v *and let* u_j, $j = 1, 2, ..., v$, *be orthonormal eigenfunctions of* $\hat{\not{D}}(\lambda_*(a))$ *with eigenvalues in* $I_a(\bar{\lambda})$. *Then, the kernel of* $\bar{\not{D}} - \bar{\lambda}I$ *has an orthonormal basis* $\{\bar{v}_j\}_{j=1,2,...,v}$ *such that*

$$\|\bar{v}_j - u_j\| < C_{25}(\lambda_{**})a, \qquad j = 1, 2, ..., v. \tag{4.3}$$

4.1. *Proof of Proposition* 4.1.

Let us write (see Lemma 3.1(i))

$$\not{D}^2 = -L - J,$$

where

$$L = (4a^2)^{-1} \sum_{\mu=1}^{d} (T_\mu - T_\mu^{-1})^2,$$

$$J = -(8a^2)^{-1} \sum_{\mu \neq \nu} \gamma^\mu \gamma^\nu [T_\mu - T_\mu^{-1}, T_\nu - T_\nu^{-1}].$$

Then we have

$$e^{-t\not{D}^2} = \sum_{k=0}^{\infty} \int_0^t d\tau_k \int_0^{\tau_k} d\tau_{k-1}$$

$$\cdots \int_0^{\tau_2} d\tau_1 \, e^{(t-\tau_k)L} J e^{(\tau_k - \tau_{k-1})L} \cdots e^{(\tau_2 - \tau_1)L} J e^{\tau_1 L}. \tag{4.4}$$

In what follows, we estimate $\mathrm{Tr}_a \, e^{-t\not{D}^2}$ by using the system of vectors $|i\rangle |y\rangle$, $i = 1, 2, ..., f$, $y \in \Lambda_a$, where the $|i\rangle$'s constitute a complete orthonormal system of a fiber of E and

$$|y\rangle = a^{-d/2} \delta_y(\cdot), \qquad y \in \Lambda_a.$$

In the above, $\delta_y(x) = 1$ (if $x = y$), $= 0$ (if $x \neq y$). Define the free translation $\dot{T}_\mu$ by

$$\dot{T}_\mu |y\rangle = |y - a\hat{\mu}\rangle, \qquad y \in \Lambda_a, \quad \mu = 1, 2, ..., d, \tag{4.5}$$

the free Laplacian $\dot{L}$ by

$$\dot{L} = (4a^2)^{-1} \sum_{\mu=1}^{d} (\dot{T}_\mu - \dot{T}_\mu^{-1})^2$$

and $\dot{J}$ by

$$\dot{J} = \sum_{\mu \neq \nu} (\dot{T}_\mu + \dot{T}_\mu^{-1})(\dot{T}_\nu + \dot{T}_\nu^{-1}).$$

Obviously (see Lemma 3.1(i)), it holds that

$$|\langle x| \langle i| J |j\rangle \, y\rangle| < C_{26} \langle x| \dot{J} |y\rangle, \qquad i, j = 1, 2, ... f; x, y \in \Lambda_a. \tag{4.6}$$

Furthermore we can easily show the Feynman–Kac formula

$$\langle x| \langle i| e^{tL} |j\rangle |y\rangle = \int_{\Omega_t(y,x)} dP_{t,y}(\omega) \langle i| T_\omega |j\rangle. \tag{4.7}$$

In the above, $\omega = \omega(\cdot)$ is a simple random walk on Λ_a with continuous time, and $\Omega_t(y, x)$ denotes the set of walks ω such that $\omega(0) = y$ and $\omega(t) = x$. The probability measure is written as $dP_{t,y}(\omega)$ and T_ω stands for the translation along the walk ω. Note that

$$|\langle x| \langle i| e^{tL} |j\rangle |y\rangle| \leqslant \int_{\Omega_t(y,x)} dP_{t,y}(\omega) = \langle x| e^{t\dot{L}} |y\rangle.$$

Then, (4.4), (4.6), and (4.7) yield

$$\mathrm{Tr}_a e^{-t\not{D}^2} < f \, \mathrm{Tr}_a \exp(t(\dot{L} + C_{27}\dot{J})). \tag{4.8}$$

The right-hand side of (4.8) can be estimated easily by means of the plane wave basis. Then the proposition follows.

4.2. *Proof of Proposition* 4.2 (*the First Step*)

The proof of Proposition 4.2 is divided into several steps. First we show the following lemma.

LEMMA 4.3. *Let D be a self-adjoint operator on a Hilbert space X with a pure point spectrum and let $\varepsilon > 0$, $\lambda \in \mathbb{R}$, and $\nu = 1, 2, ...,$. Suppose that $u_j \in X$, $j = 1, 2, ..., \nu$, satisfy*

$$\|Du_j - \lambda u_j\| \leqslant \varepsilon, \qquad j = 1, 2, ..., \nu, \tag{4.9}$$

$$\begin{aligned} |(u_i, u_j)| &\leqslant (2\nu)^{-1}, \qquad &&\text{if} \quad i \neq j, \\ &\geqslant 1 - (2\nu)^{-1}, \qquad &&\text{if} \quad i = j. \end{aligned} \tag{4.10}$$

Then, D has at least ν eigenvalues in the interval $[\lambda - (2\nu)^{1/2}\varepsilon, \lambda + (2\nu)^{1/2}\varepsilon]$.

Proof. Let $\{v_k\}$ be a complete orthonormal basis of X consisting of eigenfunctions of D and let λ_k be the eigenvalue of v_k. We expand u_j as $u_j = \sum_k \xi_{jk} v_k$, $j = 1, 2, ..., v$. Put $K = \{k \mid |\lambda_k - \lambda| \leqslant (2v)^{1/2}\varepsilon\}$. Then we obtain from (4.9)

$$\sum_{k \notin K} |\xi_{jk}|^2 < (2v)^{-1}. \tag{4.11}$$

Let us consider the vectors $\xi^{(j)} = (\varepsilon_{jk})_{k \in K}$, $j = 1, 2, ..., v$, and the matrix U defined by

$$U_{ij} = (\xi^{(i)}, \xi^{(j)}), \qquad i, j = 1, 2, ..., v.$$

Since (4.10) and (4.11) yield

$$\begin{aligned} |U_{ij}| &< v^{-1}, && \text{if} \quad i \neq j, \\ &> 1 - v^{-1}, && \text{if} \quad i = j, \end{aligned}$$

U^{-1} must exist and hence $\xi^{(j)}$, $j = 1, 2, ..., v$, turn out to be linearly independent. This means $v \leqslant |K|$ (the number of elements in K). ∎

4.3. *From Continuum to Discrete* (*the Second Step*)

Fix a positive number λ_{**} and a lattice spacing a so that the assumption of Lemma 3.4 is satisfied and choose an energy threshold $\lambda_* \in [\lambda_{**}, 2\lambda_{**}]$ that does not belong to the spectrum of $\not{D}$.

For a normalized eigenfunction $\bar{u}$ of $\bar{\not{D}}$ with an eigenvalue $\bar{\lambda} \in [-\lambda_{**}, \lambda_{**}]$, we define $\check{u}$ and $u \in L^2(\Lambda_a)$ by

$$\begin{aligned} \check{u}(x) &= \bar{u}(x), \qquad x \in \Lambda_a, \\ u &= P^+(\lambda_*)\check{u}. \end{aligned}$$

LEMMA 4.4. *If* $C_{28}a(1 + \lambda_{**})^3 N^{d+1}(1 + \Delta^{-1}) < 1$,

$$\|\hat{\not{D}}(\lambda_*)u - \bar{\lambda}u\| < C_{29}(\lambda_{**})a, \tag{4.12}$$

$$\|u - \bar{u}\| < C_{30}(\lambda_{**})\, a(1 + \Delta^{-1}) \tag{4.13}$$

hold for some constants depending on λ_{**}.

Proof. As is easily seen, we have

$$\|\not{D}\check{u} - \bar{\lambda}\check{u}\| < C_{31}(\lambda_{**})a, \tag{4.14}$$

$$\|(T_\mu + T_\mu^{-1} - 2I)\check{u}\| < C_{32}(\lambda_{**})\, a^2, \tag{4.15}$$

$$\|\bar{u} - \check{u}\| < C_{33}(\lambda_{**})a. \tag{4.16}$$

These bounds imply that the function $\tilde{u}=P(\lambda_*)\check{u}$ fulfills

$$\|\not{D}\tilde{u}-\bar{\lambda}\tilde{u}\|<C_{34}(\lambda_{**})a, \tag{4.17}$$

$$\|(T_\mu+T_\mu^{-1}-2I)\tilde{u}\|<C_{35}(\lambda_{**})\,aN(1+\Delta^{-1}), \tag{4.18}$$

$$\|\check{u}-\tilde{u}\|<C_{36}(\lambda_{**})\,a\Delta^{-1}. \tag{4.19}$$

Equations (4.17) and (4.18) follow from (4.14), (4.15), and (3.8). In order to show (4.19), we expand u as $u=\sum_j \eta_j u_j$, where $\not{D}u_j=\lambda_j u_j$, and $\|u_j\|=1$. Since (4.14) implies

$$\Delta^2\sum_{j:|\lambda_j|>\lambda_*}|\eta_j|^2<\sum_j|\eta_j|^2(\lambda_j-\bar{\lambda})^2<C_{37}(\lambda_{**})\,a^2,$$

we have (4.19). Equation (4.12) follows from (4.17) and (3.17), while (4.13) follows from (4.16) and (4.19) and

$$\|P_\mu^-\tilde{u}\|<C_{38}(\lambda_{**})\,a^2N^d(1+\Delta^{-1}),$$

which is deduced from (4.18) by the help of (3.16). ∎

4.4. *From Discrete to Continuum* (*the Third Step*)

Fix a normalized eigenfunction $u\in L^2(\Lambda_a)$ of $\hat{\not{D}}(\lambda_*)$ with an eigenvalue λ. We will construct an approximate eigenfunction $\bar{u}\in L^2(\Lambda_0)$ of $\bar{\not{D}}$ which is close to u.

Let

$$\Lambda_a^{(1)}=((2^{-1}a\mathbb{Z})\times(a\mathbb{Z})^{d-1})/(L\mathbb{Z})^d$$

be the lattice made from Λ_a by refining the x_1 axis and adding all middle points on it to Λ_a. Successively refined lattices are written as

$$\Lambda_a^{(\mu)}=((2^{-1}a\mathbb{Z})^\mu\times(a\mathbb{Z})^{d-\mu})/(L\mathbb{Z})^d, \qquad \mu=0,1,2,...,d.$$

Define the operator $R_\mu\colon L^2(\Lambda_a^{(\mu-1)})\to L^2(\Lambda_a^{(\mu)})$, $\mu=1,2,...,d$, by

$$\begin{aligned} R_\mu v(x)&=\tfrac{1}{8}(T_\mu+T_\mu^{-1}+6I)\,v(x), && \text{if}\quad x\in\Lambda_a^{(\mu-1)},\\ &=\tfrac{1}{2}(T'_\mu+T'^{-1}_\mu)\,v(x), && \text{if}\quad x\in\Lambda_a^{(\mu)}\backslash\Lambda_a^{(\mu-1)}, \end{aligned}$$

where T'_μ stands for the covariant translation by $a/2$ along the x_μ axis. (R_μ is the product of the "simple refinement" $\tilde{R}_\mu$ defined by

$$\begin{aligned} \tilde{R}_\mu v(x)&=v(x), && \text{if}\quad x\in\Lambda_a^{(\mu-1)},\\ &=\tfrac{1}{2}(T'_\mu+T'^{-1}_\mu)\,v(x), && \text{if}\quad x\in\Lambda_a^{(\mu)}\backslash\Lambda_a^{(\mu-1)}, \end{aligned}$$

and the "smoothing operator" $\frac{1}{4}(T'_\mu+T'^{-1}_\mu+2I)$.)

We put $R = R_d \cdots R_2 R_1$ and define $u^{(n)}$, $n = 0, 1, 2, \ldots$, as

$$u^{(0)} = u,$$
$$u^{(n+1)} = Ru^{(n)}, \qquad n = 0, 1, 2, \ldots,$$

by a slight abuse of notation.

LEMMA 4.5. *For $n = 0, 1, 2, \ldots$, it holds that*

$$\|u^{(n+1)} - u^{(n)}\| < C_{39} 2^{-n} a(1 + \lambda_{**})^2 N^d (1 + \Delta^{-1}), \tag{4.20}$$

$$\|(\not{D}^{(n)} - \lambda I)\, u^{(n)}\| < C_{40} a(1 + \lambda_{**})^3 N^{d+1} (1 + \Delta^{-1}), \tag{4.21}$$

$$\|(T_\mu^{(n)} + T_\mu^{(n)-1} - 2I)\, u^{(n)}\| < C_{41} (2^{-n} a)^2 (1 + \lambda_{**})^2 N^d (1 + \Delta^{-1}), \tag{4.22}$$

$$\begin{aligned} &\|2(T_\mu^{(n+1)} - I)\, u^{(n+1)} - (T_\mu^{(n)} - I)\, u^{(n)}\| \\ &\quad < C_{42} (2^{-n} a)^{3/2} (1 + \lambda_{**})^2 N^d (1 + \Delta^{-1}), \end{aligned} \tag{4.23}$$

$$|\lambda| < \lambda_* + C_{43} a^2 (1 + \lambda_{**})^3 N^{d+1} (1 + \Delta^{-1}), \tag{4.24}$$

where $T_\mu^{(n)}$ and $\not{D}^{(n)}$ are the covariant translation and the Dirac operator on the lattice $\Lambda_{2^{-n}a}$, respectively.

Proof. From the definition of R, we have

$$\begin{aligned} &\|(I - R)v\| \\ &\quad \leqslant C_{44} a \|v\| + \sum_{\mu=1}^{d} (\|(T_\mu - I)v\| + \|(T_\mu + T_\mu^{-1} - 2I)v\|), \end{aligned} \tag{4.25}$$

$$\begin{aligned} &\|(\not{D}' - \lambda I)\, Rv\| \\ &\quad \leqslant \|(\not{D} - \lambda I)v\| + C_{45} a \|v\| + a^{-1} \|(T_\mu + T_\mu^{-1} - 2I)v\|, \end{aligned} \tag{4.26}$$

$$\begin{aligned} &\|(T'_\mu + T'^{-1}_\mu - 2I)\, Rv\| \\ &\quad \leqslant 2^{-5/2} \|(T_\mu + T_\mu^{-1} - 2I)v\| + C_{46} a^2 \|v\|, \end{aligned} \tag{4.27}$$

$$\begin{aligned} &\|2(T'_\mu - I)\, Rv - (T_\mu - I)v\| \\ &\quad \leqslant C_{47} \sum_{\nu=1}^{d} \|(T_\nu + T_\nu^{-1} - 2I)v\| + \|(T'_\mu + T'^{-1}_\mu - 2I)\, Rv\| \\ &\qquad + C_{48} a \|(T_\mu - I)v\| + C_{49} a^{3/2} \|v\|, \end{aligned} \tag{4.28}$$

where $\not{D}'$ is the Dirac operator on $L^2\,(\Lambda_{a/2})$ and $v \in L^2(\Lambda_a^{(\mu-1)})$. By trivial changes of the notations, (4.25)–(4.28) hold on finer lattices.

On the other hand, our starting point is

$$\|(\not{D}-\lambda I)u\| \leqslant C_{50}a^2(1+\lambda_{**})^3 N^{d+1}(1+\Delta^{-1}), \tag{4.29}$$

$$\|(T_\mu + T_\mu^{-1} - 2I)u\| \leqslant C_{51}a^2(1+\lambda_{**})^2 N^d(1+\Delta^{-1}). \tag{4.30}$$

Equation (4.29) is obtained by using (3.17), and (4.30) follows from $P_\mu^+ u = u$ (that is, $(P_\mu^+ - P_\mu^-)u = u$) by the help of (3.16) and (3.8). Then, the successive use of (4.25)–(4.28) with a replaced by $2^{-n}a$, $n = 0, 1, 2, \ldots$, yields (4.20)–(4.23). Equation (4.24) is obtained from (4.29), since Lemma 4.3 implies the existence of an eigenvalue λ' ($\in [-\lambda_*, \lambda_*]$) of $P(\lambda_*)\not{D}$ in a certain interval containing λ. ∎

The bound (4.20) assures the existence of $\bar{u} = \lim_{n\to\infty} u^{(n)}$ in $L^2(\Lambda_0)$.

LEMMA 4.6. *If $C_{52}a(1+\lambda_{**})^2 N^d(1+\Delta^{-1}) < 1$, the function $\bar{u}$ belongs to the domain of $\bar{\not{D}}$ and satisfies*

$$\|u - \bar{u}\| < C_{53}a(1+\lambda_{**})^2 N^d(1+\Delta^{-1}), \tag{4.31}$$

$$\|(\bar{\not{D}} - \lambda I)\bar{u}\| < C_{54}a(1+\lambda_{**})^3 N^{d+1}(1+\Delta^{-1}). \tag{4.32}$$

Proof. Equation (4.31) is obvious from (4.20). Consider the operator $E\colon L^2(\Lambda_a) \to L^2(\Lambda_0)$ defined by

$$(Ev)(x + a\xi) = \sum_{\mu=1}^{d} ((1-\xi_\mu)I + \xi_\mu \dot{T}_\mu)\, v(x),$$
$$v \in L^2(\Lambda_a),\ x \in \Lambda_a,\ \xi = (\xi_\mu) \in [0, 1)^d,$$

where $\dot{T}_\mu$ is the free translation defined by (4.5) through the local trivialization fixed earlier. Then, we have

$$\|(E - I)v\| \leqslant \sum_{\mu=1}^{d} \|(T_\mu - I)v\| + C_{55}a\,\|v\|, \tag{4.33}$$

$$\begin{aligned}\|(\partial_\mu + iA_\mu)\, Ev - a^{-1}(T_\mu - I)v\| \leqslant{}& C_{56}a^2\,\|v\| + 2d \sum_{\nu=1}^{d} \|(T_\nu + T_\nu^{-1} - 2I)v\| \\ &+ C_{57}a \sum_{\nu=1}^{d} \|(T_\nu - I)v\|.\end{aligned} \tag{4.34}$$

Note that these inequalities hold on finer lattices by a trivial change of notations.

Consider the sequence $Eu^{(n)} \in L^2(\Lambda_0)$, $n = 0, 1, 2, \ldots$. Then, (4.33), together with (4.20) and (4.23), show $\bar{u} = \lim_{n\to\infty} Eu^{(n)}$. Furthermore, (4.34) and (4.23) imply that the sequence $\{\bar{\not{D}}Eu^{(n)}\}$ converges in $L^2(\Lambda_0)$.

Since $\bar{\not{D}}$ is a closed operator, $\bar{u}$ belongs to the domain of $\bar{\not{D}}$ and

$$\bar{\not{D}}\bar{u} = \lim_{n \to \infty} \bar{\not{D}} E u^{(n)}$$

holds. Equation (4.32) follows from (4.21), (4.34), and (4.22). ∎

4.5. *Completion of the Proof of Proposition* 4.2

(i) Let $\check{\lambda}$ be the smallest eigenvalue of $\bar{\not{D}}$ in the interval (λ_{**}, ∞) and put $\hat{\lambda} = \min(\check{\lambda}, 2\lambda_{**})$. By virtue of Lemma 4.3, we can choose $\lambda_*(a) \in [(2\lambda_{**} + \hat{\lambda})/3, (\lambda_{**} + 2\hat{\lambda})/3]$ so that (4.1) and (4.2) hold.

(ii) Combining Lemmas 4.6 and 4.3, we see that, for each eigenvalue $\hat{\lambda}$ of $\hat{\not{D}}(\lambda_*(a))$, there exists an eigenvalue $\bar{\lambda}$ of $\bar{\not{D}}$ close to $\hat{\lambda}$. Furthermore, using (4.24) with $\lambda_* = \lambda_*(a)$, we see that, if a is sufficiently small, each eigenvalue of $\hat{\not{D}}(\lambda_*(a))$ belongs to one of the disjoint intervals $I_a(\bar{\lambda})$, where $\bar{\lambda}$ is an eigenvalue of $\bar{\not{D}}$ such that $|\bar{\lambda}| \leqslant \lambda_{**}$. Let v be the multiplicity of the eigenvalue $\bar{\lambda} \in [-\lambda_{**}, \lambda_{**}]$ of $\bar{\not{D}}$. Then as is seen from Lemma 4.3, 4.4, and 4.6, the interval $I_a(\bar{\lambda})$ contains just v eigenvalues of $\hat{\not{D}}(\lambda_*(a))$.

(iii) Let λ_j be the eigenvalue of $\hat{\not{D}}(\lambda_*(a))$ corresponding to u_j. According to the procedure described in Lemma 4.6, we can find approximate eigenfunctions $\bar{u}_j$ of $\bar{\not{D}}$ close to u_j. Expand $\bar{u}_j$ as

$$\bar{u}_j = \sum_n \xi_{jn} \bar{w}_n, \qquad j = 1, 2, ..., v,$$

by means of normalized eigenfunctions $\bar{w}_n$ of $\bar{\not{D}}$:

$$\bar{\not{D}} \bar{w}_n = \bar{\lambda}_n \bar{w}_n.$$

Put $K = \{n \mid \bar{\lambda}_n = \bar{\lambda}\}$. Then, (4.31) and (4.32) with λ, u, and $\bar{u}$ replaced by λ_j, u_j, $\bar{u}_j$, respectively, imply

$$\left\| u_j - \sum_{n \in K} \xi_{jn} \bar{w}_n \right\| < C_{58}(\lambda_{**}) a, \qquad j = 1, 2, ..., v.$$

We define the $v \times v$ matrix $\Xi = (\xi_{jn})_{1 \leqslant j \leqslant v, n \in K}$. Since Ξ is approximately unitary, there exists a unitary matrix $\Xi' = (\xi'_{jn})_{1 \leqslant j \leqslant v, n \in K}$ such that

$$|\xi_{jn} - \xi'_{jn}| < C_{59}(\lambda_{**}) a, \qquad j = 1, 2, ..., v,\ n \in K.$$

Putting

$$\bar{v}_j = \sum_{n \in K} \xi'_{jn} \bar{w}_n, \qquad j = 1, 2, ..., v,$$

we obtain an orthonormal basis $\{\bar{v}_j\}$ of $\operatorname{Ker}(\bar{\not{D}} - \bar{\lambda} I)$ satisfying (4.3)

5. Continuum Limit

We assume that a satisfies the assumption of Lemma 3.4.

5.1. *Berezin Integral*

Performing the Berezin integral in $\langle Y_t(x)\rangle_a$, we obtain [13, 28]

$$\begin{aligned} a^d \sum_{x\in\Lambda_a} &\xi(x)\langle Y_t(x)\rangle_a \\ &= -i\,\mathrm{Tr}_a\left[e^{-t\not{D}^2}\left\{\left(i\not{D}+MI-\frac{r}{2a}e^{i\theta\gamma^5}W\right)^{-1}, (i\not{D}+MI)\right\}\gamma^5\xi\right] \end{aligned} \tag{5.1}$$

for a smooth function $\xi(x)$ on Λ_0. We have denoted the trace on $L^2(\Lambda_a)$ by Tr_a and the anticommutator by $\{\ ,\ \}$ and have regarded the function $\xi(x)$ as a multiplying operator on $L^2(\Lambda_a)$. Let us put

$$H_0 = i\not{D} + MI,$$

$$H_1 = i\not{D} + MI - \frac{r}{2a}e^{i\theta\gamma^5}W,$$

$$\Gamma = \gamma^5\xi.$$

Then, (5.1) can be written as

$$a^d \sum_{x\in\Lambda_a} \xi(x)\langle Y_t(x)\rangle_a = -i\,\mathrm{Tr}_a[e^{-t\not{D}^2}\{H_1^{-1}, H_0\}\Gamma]. \tag{5.2}$$

Remarks. 1. For a hermitian matrix D, a positive number M, and a semi-positive operator V, it holds that

$$\begin{aligned} &\|(iD+MI+V)u\|^2 \\ &\quad = \|Du\|^2 + \|Mu\|^2 + \|Vu\|^2 + 2M(u, Vu) - 2\,\mathrm{Im}(Vu, Du) \\ &\quad \geqslant M^2\|u\|^2. \end{aligned}$$

Therefore, we have

$$\|H_1 u\| \geqslant M\|u\|, \qquad u\in L^2(\Lambda_a), \tag{5.3}$$

since $-W = \sum_{\mu=1}^d (T_\mu - I)^*(T_\mu - I)$ is a semi-positive operator and $(r/a)\cos\theta > 0$. Consequently, H_1 turns out to be invertible and we can define the expectation $\langle\ \rangle_a$.

2. If we could consider $\lim_{a \to 0} L^2(\Lambda_a)$, the space $L^2(\Lambda_0)$ would be contained in $\lim_{a \to 0} L^2(\Lambda_a)$ as a *proper* subspace (see Proposition 4.2). Our aim is, in some sense, to show that $\lim_{a \to 0} \frac{1}{2}\{H_1^{-1}, H_0\}$ gives the projection onto $L^2(\Lambda_0)$.

5.2. *Suppression of Species Doubling*

Let us put

$$E_0(t, a) = \mathrm{Tr}_a[e^{-t\not{D}^2}\{H_1^{-1}, H_0\}\Gamma], \tag{5.4}$$

and

$$E_1(t, \lambda_*, a) = \mathrm{Tr}_a[e^{-t\not{D}^2}P(\lambda_*)\{H_1^{-1}, H_0\}\Gamma]. \tag{5.5}$$

As is shown in the following proposition, E_0 is close to E_1 uniformly in the lattice spacing a when λ_* is large.

PROPOSITION 5.1. *For* $0 < a < 1$, *it holds that*

$$|E_0(t, a) - E_1(t, \lambda_*, a)| < C_{60} e^{-(t/2)\lambda_*^2}(1 + M^{-1})\, t^{-1/2}\chi(t/4), \tag{5.6}$$

where χ *is the function introduced in Proposition* 4.1.

Proof. We note that

$$\{H_1^{-1}, H_0\}\Gamma = \{H_1^{-1}\Gamma, H_0\} + H_1^{-1}[H_0, \Gamma],$$

and

$$[H_0, \Gamma] = -2i\Gamma\not{D} - i\gamma^5[\not{D}, \xi]. \tag{5.7}$$

$\|\Gamma\|$ and $\|[\not{D}, \xi]\|$ are bounded by some constants depending only on ξ, and $\|H_1^{-1}\|$ is bounded by M^{-1} (see (5.3)). Then, evaluating the trace by means of the eigenfunctions of $\not{D}$, we have

$$\begin{aligned}|E_0 - E_1| &< C_{61}(1 + M^{-1}) \sum_{n: |\lambda_n| > \lambda_*} (1 + |\lambda_n|)\, e^{-t\lambda_n^2} \\ &< C_{62}(1 + M^{-1})\, e^{-(t/2)\lambda_*^2} t^{-1/2} \sum_n e^{-(t/4)\lambda_n^2}. \quad \blacksquare\end{aligned}$$

We now define

$$H_2 = i \sum_\sigma P^\sigma \not{D} P^\sigma + MI + \frac{2r}{a} e^{i\theta\gamma^5} \sum_{\mu=1}^{d} P_\mu^-,$$

where $P^\sigma = \prod_{\mu=1}^{d} P_\mu^{\sigma_\mu}$, $\sigma = (\sigma_\mu)_{\mu=1,2,\dots,d}$, $\sigma_\mu = +$ or $-$, and $P_\mu^\pm$ is defined in Lemma 3.4. By a method similar to that used in deriving (5.3), it is shown that $\|H_2^{-1}\| \leqslant M^{-1}$. Put

$$E_2(t, \lambda_*, a) = \mathrm{Tr}_a[e^{-t\not{D}^2} P(\lambda_*)\{H_2^{-1}, H_0\}\Gamma]. \tag{5.8}$$

The difference between E_1 and E_2 turns out to be small when a is small:

PROPOSITION 5.2. *It holds that*

$$|E_1(t, \lambda_*, a) - E_2(t, \lambda_*, a)| < C_{63} a(1+r)(1+M^{-1})^2(1+\lambda_*)^3 N^{d+2}(1+\Delta^{-1}). \tag{5.9}$$

Proof. We must estimate

$$E_1 - E_2 = \mathrm{Tr}_a[Pe^{-t\not{D}^2}\{\Omega, H_0\}\Gamma], \tag{5.10}$$

where $P = P(\lambda_*)$ and

$$\Omega = P(H_1^{-1} - H_2^{-1}) = H_2^{-1} P(H_2 - H_1) H_1^{-1}.$$

Note that

$$P(H_2 - H_1) = i \sum_{\sigma \neq \tau} P^\sigma \not{D} P^\tau - \frac{r}{2a} e^{i\theta\gamma^5} \left(PWQ + 2 \sum_{\mu=1}^{d} (S_\mu - P_\mu^+ + P_\mu^-) \right),$$

where $Q = I - P$ and S_μ is defined by (3.11). We can estimate the right-hand side by using (3.17), (3.10) , and (3.16):

$$\|P(H_2 - H_1)\| < C_{64} a(1+r) N^{d+1}(1+\Delta^{-1})(1+\lambda_*)^3.$$

Then we have

$$\|\Omega\| < C_{65} a(1+r) M^{-2} N^{d+1}(1+\Delta^{-1})(1+\lambda_*)^3.$$

Using the above bound, we can estimate the right-hand side of (5.10) in a manner similar to that of the proof of Proposition 5.1. ∎

Put

$$E_3(t, \lambda_*, a) = 2\,\mathrm{Tr}_a[e^{-t\not{D}^2} P^+(\lambda_*)\Gamma]$$

and

$$E_4(t, \lambda_*, a) = 2\,\mathrm{Tr}_a[e^{-t(P^+(\lambda_*)\not{D}P^+(\lambda_*))^2} P^+(\lambda_*)\Gamma].$$

The following proposition shows that the Wilson term suppresses the species doubling in a nonperturbative way.

PROPOSITION 5.3. *It holds that*

$$|E_2(t, \lambda_*, a) - E_3(t, \lambda_*, a)| < C_{66} a(1 + (2r\cos\theta)^{-1})(1 + M)(1 + \lambda_*)^3 N^{d+2}(1 + \Delta^{-1}). \tag{5.11}$$

$$|E_3(t, \lambda_*, a) - E_4(t, \lambda_*, a)| < C_{67} a^2 (1 + \lambda_*)^4 N^{d+2}(1 + \Delta^{-1}) t. \tag{5.12}$$

Proof. In order to show (5.11), it suffices to estimate

$$\Omega = PH_0 H_2^{-1} - P^+,$$
$$\Omega^* = PH_2^{-1} H_0 - P^+,$$

where $P = P(\lambda_*)$ and $P^+ = P^+(\lambda_*)$. Noting that P^+ commutes with H_2 and P, we obtain

$$\Omega = PH_0 H_2^{-1}(I - P^+)P + i(I - P^+)[P\not{D}, \bar{P}^+]\, H_2^{-1}.$$

Let us estimate $\|H_2^{-1}(I - P^+)P\|$. We can restrict our attention to the subspace $\mathrm{Im}(I - P^+)P$, which is the direct sum of $\mathrm{Im}\, P^\sigma$ such that $\sigma_\mu = -$ for some $\mu = 1, 2, ..., d$. Let $m(\sigma)$ be the number of μ's such that $\sigma_\mu = -$. Since $H_2^{-1}(I - P^+)P$, restricted to $\mathrm{Im}\, P^\sigma$, is equal to H_2^{-1} and

$$H_2|_{\mathrm{Im}\, P^\sigma} = iP^\sigma \not{D} P^\sigma + M + \frac{2r}{a} e^{i\theta\gamma^5} m(\sigma),$$

we obtain

$$\|H_2|_{\mathrm{Im}\, P^\sigma}\| > \frac{2r}{a}\cos\theta$$

in a way similar to that in which we obtained (5.3). This proves

$$\|H_2^{-1}(I - P^+)P\| < (2r\cos\theta)^{-1} a.$$

The above bound, together with (3.17), yields

$$\|\Omega\| = \|\Omega^*\| < C_{68} a(1 + 2r\cos\theta)^{-1})(1 + M)(1 + \lambda_*)^3 N^{d+1}(1 + \Delta^{-1}).$$

Let us show (5.12). Put

$$A = tP\not{D}P\not{D}P,$$
$$B = tP^{+}\not{D}P^{+}\not{D}P^{+}.$$

Since A and B are semi-positive and $[B, P^{+}] = 0$, we have

$$\begin{aligned}|\mathrm{Tr}_a((e^{-A} - e^{-B})\, P^{+}\Gamma)| &\leqslant N\|(e^{-A} - e^{-B})\, P^{+}\|\, \|\Gamma\| \\ &\leqslant N\|(A - B)\, P^{+}\|\, \|\Gamma\| \\ &\leqslant C_{69} a^2 (1 + \lambda_*)^4 N^{d+2} (1 + \Delta^{-1})\, t,\end{aligned}$$

where we have used the formula

$$e^{-A} - e^{-B} = \int_0^1 ds\, e^{-(1-s)A}(A - B)\, e^{-sB}. \quad \blacksquare$$

5.3. *Continuum Limit*

By the help of Proposition 4.2, we can take the continuum limit. Let

$$\bar{\not{D}} = \sum_n \bar{\lambda}_n \bar{P}_n$$

be the spectral decomposition of $\bar{\not{D}}$ and put

$$\bar{P}(\lambda_*) = \sum_{n : |\lambda_n| \leqslant \lambda_{**}} \bar{P}_n.$$

PROPOSITION 5.4. *It holds that*

$$\lim_{a \to 0} E_4(t, \lambda_*(a), a) = 2\, \mathrm{Tr}(e^{-t\bar{\not{D}}^2} \bar{P}(\lambda_{**})\Gamma), \tag{5.13}$$

where Tr *denotes the trace on* $L^2(\Lambda_0)$.

Combining (5.2), (5.4), (5.6), (5.9), (5.11), (5.12), and (5.13), we obtain the following theorem:

THEOREM 5.5. *It holds that*

$$\lim_{a \to 0} a^d \sum_{x \in \Lambda_a} \xi(x) \langle Y_t(x) \rangle_a = -2i\, \mathrm{Tr}(e^{-t\bar{\not{D}}^2} \gamma^5 \xi). \tag{5.14}$$

The limit $t \to 0$ of the right-hand side of (5.14) can be taken by means of the local index theorem [24, 25]. As a result, we obtain Theorem in Section 2.

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